

## Entanglement through conformal interfaces

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**ABSTRACT:** We consider entanglement through permeable interfaces. We compute the partition functions with the interfaces inserted. By the replica trick, the entanglement entropy is obtained analytically. The entropy scales logarithmically with respect to the size of the system, similarly to the universal scaling of the ordinary entanglement entropy in  $(1+1)$ -dimensional conformal field theory. Its coefficient, however, is not constant but controlled by the permeability, the dependence on which is expressed through the dilogarithm function. The sub-leading term of the entropy counts the winding numbers, showing an analogy to the topological entanglement entropy which characterizes the topological order in  $(2+1)$ -dimensional systems.

**KEYWORDS:** Field Theories in Lower Dimensions, Conformal and W Symmetry, Boundary Quantum Field Theory, Conformal Field Models in String Theory.

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## 1. Introduction

Conformal interfaces provide a natural framework to extend the (1+1)-dimensional conformal field theory (CFT) with boundaries. Taking into account the important roles played by the boundary CFT in condensed matter physics and string theory, one expects that the interface CFT also opens up new directions in such studies. In fact, interesting properties have already been found: For example, all the symmetries of the rational CFT are generated by a class of interfaces called topological interfaces [1]. Topological interfaces also transform one set of D-branes to another [2, 3]. For (potential) applications and a (partial) list of references, we refer to [1–13] and references therein.

When a system consists of two (or more) sub-systems as in the interface CFT, the quantum correlation, i.e., entanglement, between the sub-systems is a useful probe to the system. The entanglement entropy is a measure of this entanglement. In (1+1)-dimensional systems, the entanglement entropy of the ground state shows a sharp contrast between the critical and the non-critical regime [14] and, at the critical point, there appears a universal logarithmic scaling with respect to the size of the system characterized by the central charge [15, 16]. In (2+1)-dimensional systems, the leading term of the ground state entanglement entropy scales linearly as the boundary size of the system, whereas the sub-leading term characterizes the topological order of the systems, named as the topological entanglement entropy [17, 18]. Entanglement is essential also in quantum computation and

information. The references on the entanglement entropy in (1+1)-dimensional systems with defects include [19–21].

In this note, we consider the entanglement entropy in the  $c = 1$  interface CFT. A class of conformal interfaces in this theory has been constructed which interpolates perfectly transmitting and reflecting interfaces [6]. In addition to the physical relevance of the  $c = 1$  CFT, because of the fact that these permeable interfaces are simple but possessing structures characterized by some parameters, we expect that they provide useful insights into the interface CFT. The entanglement entropy may be a useful probe, and its role in the interface CFT is of interest.

In section 2, we briefly summarize the entanglement entropy with slight generalizations in the case of the interface CFT. We also introduce the  $c = 1$  permeable interfaces. In section 3, we compute the partition functions with the interfaces inserted. In section 4, using this result and the replica trick, we obtain the entanglement entropy analytically when the sizes of the two CFT's joining at the interface are equal. The entropy scales logarithmically with respect to the size of the system, similarly to the universal scaling in the case without interfaces. However, its coefficient is not constant but controlled by the permeability. The sub-leading term counts the product of the winding numbers. This shows an analogy to the topological entanglement entropy in (2+1)-dimensional systems. In deriving the entropy, we adopt two approaches: One is based on the Bernoulli polynomials and numbers, which provides a general method for carrying out the replica trick. Another is a direct evaluation of a sum by an integral for the large size of the system. In the course of verifying the equivalence of the results from the two approaches, we find that the scaling coefficient is expressed by the dilogarithm function. In section 5, we conclude with a brief summary. Some useful formulas are collected in the appendix. The entanglement entropy in the  $c = 1$  interface CFT has been discussed in [22] in a different setting from ours and in the context of the boundary entropy and its holographic dual.

## 2. Setup

### 2.1 Entanglement entropy in interface CFT

We consider two (1+1)-dimensional CFT's defined on a half complex plane  $\text{Re } w > 0$  and  $\text{Re } w < 0$ , respectively. The interface between CFT<sub>1</sub> and CFT<sub>2</sub> lies along the imaginary axis  $\text{Re } w = 0$ . The conformal invariance requires the continuity condition

$$L_n^1 - \tilde{L}_{-n}^1 = L_n^2 - \tilde{L}_{-n}^2, \tag{2.1}$$

at the interface, where  $L_n^a, \tilde{L}_n^a$  ( $a = 1, 2; n \in \mathbf{Z}$ ) are the left and the right Virasoro generators of CFT<sub>a</sub>.

We are interested in the entanglement entropy of the ground state. The entropy is defined by the von Neumann entropy of the reduced density matrix for the ground state  $\rho_1 = \text{Tr}_2 |0\rangle\langle 0|$  as

$$S = -\text{Tr}_1 \rho_1 \log \rho_1 = -\left. \frac{\partial}{\partial K} \text{Tr}_1 \rho_1^K \right|_{K=1}, \tag{2.2}$$

where  $\text{Tr}_a$  stands for the trace over the degrees of freedom in  $\text{CFT}_a$ . The trace of the  $K$ -th power of the reduced density matrix is represented by a partition function on a  $K$ -sheeted Riemann surface  $\mathcal{R}_K$  whose branch cut runs along the real axis from  $w = 0$  to  $\infty$  [23]:

$$\text{Tr}_1 \rho_1^K = \frac{Z(K)}{Z^K(1)} \equiv \frac{1}{Z^K(1)} \int \mathcal{D}\phi \exp \left[ - \int_{\mathcal{R}_K} d^2w \mathcal{L}(\phi) \right], \quad (2.3)$$

where  $\phi$  represents the fields of  $\text{CFT}_1$  and  $\text{CFT}_2$ . The interface is inserted on each sheet of  $\mathcal{R}_K$ . The normalization factor  $1/Z^K(1)$  assures  $\text{Tr}_1 \rho_1 = 1$ . From this path-integral representation, the entropy is given by

$$S = (1 - \partial_K) \log Z(K) \Big|_{K=1}. \quad (2.4)$$

As in the case without interfaces, how mixed the reduced density matrix is depends on the correlation across the interface, and hence the von Neumann entropy measures the entanglement of the CFT's through the interface.

To evaluate the partition function with the interface inserted, we move to  $z = \log w$  plane. Introducing cutoffs at  $|w| = \epsilon$  and  $|w| = L$  [15, 19], the  $K$ -sheeted Riemann surface  $\mathcal{R}_K$  is mapped to a rectangular whose lengths along the real and the imaginary axis are  $(\log(L/\epsilon), 2\pi K)$ , respectively. The interface is mapped to  $\text{Im } z = (2m - 1)\pi/2$  ( $m = 1, \dots, 2K$ ). In the following, we set  $\epsilon = 1/L$  for simplicity.

Here, we impose the periodic boundary condition along the  $\text{Re } z$  direction. When the interface is absent, this reduces to the treatment in [15], and we can check that our final results actually reproduce those in [15] as a special case. (Our  $L$  corresponds to  $\Sigma/\epsilon$  in [15], as is clear from our setting.) Also, when the interface is topological (see below), our partition function is regarded as a generalization (in the  $c = 1$  case) of the generalized twisted partition functions discussed in [5]. In order to carry out the analysis independently of the boundary conditions, it is desirable to generalize the analysis in [15, 16, 24] based on the conformal invariance to our case with interfaces.

Now, our partition function is a torus partition function with  $2K$  interfaces inserted. Recall that the ground state density matrix is represented by the product of the ground state wave function  $\Psi_0 \Psi_0^*$  in deriving the path-integral representation (2.3), and that  $\Psi_0$  ( $\Psi_0^*$ ) gives the path-integral on the lower (upper) half  $w$ -plane. One then finds that the interfaces at  $\text{Im } z = (2m - 1)\pi/2$  for odd  $m$  and those for even  $m$  are hermitian conjugate with each other. Therefore, we find that

$$Z(K) = \text{Tr}_1 \left( I_{12} q^{L_0^2 + \bar{L}_0^2} (I_{12})^\dagger q^{L_0^1 + \bar{L}_0^1} I_{12} \dots (I_{12})^\dagger q^{L_0^1 + \bar{L}_0^1} \right), \quad (2.5)$$

where  $q = e^{-2\pi t}$  with  $t = \pi/(2 \log L)$ , we have denoted the interfaces by  $I_{12}$  and  $(I_{12})^\dagger$ , and each of them appears  $K$  times alternately. We have also rescaled the rectangular in the  $z$ -plane so that it becomes of the standard lengths  $(2\pi, 2\pi t)$ . The subscript ‘‘12’’ is put to make explicit the fact that  $I_{12}$  joins  $\text{CFT}_1$  and  $\text{CFT}_2$  in this order. The above expression is manifestly symmetric with respect to  $\text{CFT}_1$  and  $\text{CFT}_2$ , as it should be.

## 2.2 $c = 1$ permeable interfaces

We now specialize our discussion to the case of the  $c = 1$  permeable interfaces. A general way to construct conformal interfaces is to use the folding trick [4]: Let us set  $\tilde{\tau} = \text{Re } w$ . Then, by flipping the sign of  $\tilde{\tau}$  for  $\tilde{\tau} < 0$ ,  $\text{CFT}_2$  comes to live on the  $\tilde{\tau} > 0$  half plane. In the course, the interface becomes a conformal boundary, which can be expressed by a conformal boundary state of the tensor product theory  $\text{CFT}_1 \otimes \text{CFT}_2$ . In fact, the condition of the conformal invariance of the interface (2.1) becomes that of the conformal boundary,  $L_n^1 + L_n^2 - (\tilde{L}_{-n}^1 + \tilde{L}_{-n}^2) = 0$ , since the left and the right movers and the positive and the negative modes in  $\text{CFT}_2$  are exchanged, respectively, by the folding. Conversely, unfolding a conformal boundary state gives a conformal interface.

In this way, from the boundary states of the tensor product theory of two free bosons, the  $c = 1$  permeable interfaces are obtained [6]:

$$\begin{aligned}
 I_{12}^{\pm(\alpha,\beta)}(\theta_{\pm}) &= G_{12}^{\pm(\alpha,\beta)}(\theta_{\pm}) \prod_{n=1}^{\infty} e^{\frac{1}{n} \left( S_{11}^{\pm} \alpha_{-n}^1 \tilde{\alpha}_{-n}^1 - S_{12}^{\pm} \alpha_{-n}^1 \alpha_n^2 - S_{21}^{\pm} \tilde{\alpha}_{-n}^1 \tilde{\alpha}_n^2 + S_{22}^{\pm} \alpha_n^2 \tilde{\alpha}_n^2 \right)}, \\
 G_{12}^{+(\alpha,\beta)}(\theta_+) &= g_+ \sum_{N,M=-\infty}^{\infty} e^{i(N\alpha - M\beta)} |k_2 N, k_1 M\rangle \langle k_1 N, k_2 M|, \\
 G_{12}^{-(\alpha,\beta)}(\theta_-) &= g_- \sum_{N,M=-\infty}^{\infty} e^{i(N\alpha - M\beta)} |k_1 M, k_2 N\rangle \langle k_1 N, k_2 M|,
 \end{aligned} \tag{2.6}$$

where

$$S^{\pm} = \begin{pmatrix} \mp \cos 2\theta_{\pm} & -\sin 2\theta_{\pm} \\ \mp \sin 2\theta_{\pm} & \cos 2\theta_{\pm} \end{pmatrix}, \quad g_{\pm} = \left| \frac{k_1 k_2}{\sin 2\theta_{\pm}} \right|^{1/2}, \tag{2.7}$$

and

$$\tan \theta_+ = \frac{k_2 R_2}{k_1 R_1}, \quad \tan \theta_- = \frac{2k_2 R_1 R_2}{k_1}. \tag{2.8}$$

$\alpha_n^a, \tilde{\alpha}_n^a$  ( $a = 1, 2; n \in \mathbf{Z}$ ) are the modes of the free boson  $\phi_a$  compactified on a circle with radius  $R_a$ . They satisfy  $[\alpha_m^a, \alpha_n^b] = m \delta_{m,n} \delta^{ab}$  and similar expressions for  $\tilde{\alpha}_n^a$ . It is understood that  $\alpha_{-n}^a, \tilde{\alpha}_{-n}^a$  ( $n > 0$ ) implicitly act on  $G_{12}^{\pm}$  from the left and  $\alpha_n^a, \tilde{\alpha}_n^a$  ( $n > 0$ ) from the right.  $k_a$  is the winding number for  $\phi_a$ .  $|n_a, m_a\rangle$  is the oscillator vacuum for  $\phi_a$  with the momentum  $n_a/R_a$  and the winding number  $m_a$ . Its dual is denoted by  $\langle n_a, m_a|$ .  $I_{12}^+$  is obtained from the boundary state with one Dirichlet and one Neumann boundary condition.  $\theta_+$  is the angle between the  $\phi_1$  and the Neumann direction in the target space.  $I_{12}^-$  is obtained by T-dualizing the boundary state for  $I_{12}^+$ . In unfolding, there is a choice of which CFT is unfolded. Changing this choice gives ‘‘anti-interfaces’’  $\tilde{I}_{21}^{\pm}$  [11], which are equivalent to  $(I_{12}^+)^{\dagger}$  with some signs of the parameters flipped. One can check explicitly that the continuity condition (2.1) is satisfied by these interfaces.

The matrices  $S^{\pm}$  control interactions between  $\text{CFT}_1$  and  $\text{CFT}_2$ . When  $\theta_{\pm}$  are a multiple of  $\pi/2$ , the two CFT’s decouple and the interfaces become totally reflective. When  $\theta_{\pm}$  are odd multiple of  $\pi/4$ , the interfaces become totally transmissive. In this case, the

interface is called topological, since each of the left and the right energy-momentum tensor becomes continuous across the interface, i.e.,  $L_n^1 = L_n^2$  and  $\tilde{L}_n^1 = \tilde{L}_n^2$ , and hence the interface can be freely deformed. Note that the identity operator is included as a special case of the topological interface  $I_{12}^{+(0,0)}(\pi/4)$ .

### 3. Partition functions with interfaces inserted

In this section, we compute the partition function with the interfaces inserted, i.e.,  $Z(K)$  in (2.5) for  $I_{12}^\pm$ . The computation in the following can be regarded as a generalization of (part of) that in [11] for the fusion of the interfaces.

#### 3.1 Case of $I_{12}^+$

Here, we consider the case of  $I_{12}^+$ . To carry out the computation, it is convenient to first focus on a unit of the products of the operators in  $Z(K)$ ,

$$J = I_{12}^+ q^{L_0^2 + \tilde{L}_0^2} (I_{12}^+)^\dagger q^{L_0^1 + \tilde{L}_0^1}, \quad (3.1)$$

and rewrite the quadratic oscillator parts in  $I_{12}^+$  and  $(I_{12}^+)^\dagger$  as

$$(\alpha_{-n}^1, \tilde{\alpha}_n^2) \cdot \begin{pmatrix} -c & s \\ s & c \end{pmatrix} \cdot \begin{pmatrix} \tilde{\alpha}_{-n}^1 \\ \alpha_n^2 \end{pmatrix}, \quad (\alpha_{-n}^2, \tilde{\alpha}_n^1) \cdot \begin{pmatrix} c & s \\ s & -c \end{pmatrix} \cdot \begin{pmatrix} \tilde{\alpha}_{-n}^2 \\ \alpha_n^1 \end{pmatrix}, \quad (3.2)$$

respectively, where  $c = \cos 2\theta_+$ ,  $s = \sin 2\theta_+$ . We then linearize the quadratic forms by an identity

$$e^{\vec{A} \cdot \vec{B}} = \int \frac{d^2 \vec{z}}{\pi^2} e^{-\vec{z} \cdot \vec{z} - \vec{z} \cdot \vec{A} - \vec{z} \cdot \vec{B}}, \quad (3.3)$$

which is valid when all  $A_i, B_i$  ( $i = 1, 2$ ) are commuting with each other. After the linearization, one can explicitly put the creation operators  $\alpha_{-n}^a, \tilde{\alpha}_{-n}^a$  ( $n > 0$ ) on the left of  $G_{12}^+$  or  $(G_{12}^+)^\dagger$ , and the annihilation operators  $\alpha_n^a, \tilde{\alpha}_n^a$  ( $n > 0$ ) on the right. Further pushing the Virasoro generators to the oscillator ground states in  $G_{12}^+$  or  $(G_{12}^+)^\dagger$  using  $e^{\alpha_n^a} q^{L_0^a} = q^{L_0^a} e^{q^n \alpha_n^a}$ , and commuting the creation and annihilation operators between  $G_{12}^+$  and  $(G_{12}^+)^\dagger$ , one finds that

$$J = \prod_n \int \frac{d^2 \vec{z}_n}{\pi^2} \int \frac{d^2 \vec{w}_n}{\pi^2} e^{-\vec{z}_n \cdot \vec{z}_n - \vec{w}_n \cdot \vec{w}_n} \times e^{q^n z_{n2} (c\bar{w}_{n1} + s\bar{w}_{n2}) + q^n (s\bar{z}_{n1} + c\bar{z}_{n2}) w_{n1}} \\ \times \prod_n e^{-\frac{1}{n} z_{n1} \alpha_{-n}^1 + (c\bar{z}_{n1} - s\bar{z}_{n2}) \tilde{\alpha}_{-n}^1} \cdot G' \cdot \prod_n e^{-\frac{1}{n} w_{n2} q^n \tilde{\alpha}_n^1 + (-s\bar{w}_{n1} + c\bar{w}_{n2}) q^n \alpha_n^1}, \quad (3.4)$$

where  $z_{ni}$  are the components of  $\vec{z}_n$  etc., and

$$G' = g_+^2 \sum_{N, M} q^{\epsilon_{k_2 N, k_1 M}^{R_1} + \epsilon_{k_1 N, k_2 M}^{R_2} - \frac{1}{6} |k_2 N, k_1 M\rangle \langle k_2 N, k_1 M|}, \quad (3.5)$$

for  $k_1 k_2 \neq 0$  with

$$\epsilon_{n, m}^R = \left( \frac{n}{2R} \right)^2 + (mR)^2. \quad (3.6)$$

For  $k_1 k_2 = 0$ , we have different expressions of  $G'$  due to the change of the zero-mode structure in  $G_{12}^+$ . We separately discuss this case later. We then take the  $K$ -th power of  $J$ , commute the creation and annihilation operators, and perform the  $z_{n2}$ - and  $w_{n1}$ -integrals so as to maintain the linearity of the oscillators. Relabeling  $z_{n1}, w_{n2}$  as  $z_n, w_n$ , we find that

$$Z(K) = \text{Tr}_1 J^K = g_+^{2K} \sum_{N,M=-\infty}^{\infty} q^{K(\epsilon_{k_2 N, k_1 M}^{R_1} + \epsilon_{k_1 N, k_2 M}^{R_2})} q^{-K/6} \prod_{n=1}^{\infty} P_n, \quad (3.7)$$

where

$$P_n = D_n^K \prod_{k=1}^K \int \frac{d^2 z_n^{(k)}}{\pi} \int \frac{d^2 w_n^{(k)}}{\pi} e^{-z_n^{(k)} \bar{z}_n^{(k)} - w_n^{(k)} \bar{w}_n^{(k)}} \times e^{s^2 q^{2n} D_n (w_n^{(k)} \bar{w}_n^{(k+1)} + \bar{z}_n^{(k)} z_n^{(k+1)}) - c q^n (1 - q^{2n}) D_n (w_n^{(k)} \bar{z}_n^{(k+1)} + \bar{w}_n^{(k)} z_n^{(k+1)})}, \quad (3.8)$$

with  $D_n = (1 - c^2 q^{2n})^{-1}$  and  $z_n^{(K+1)} = z_n^{(1)}$ ,  $w_n^{(K+1)} = w_n^{(1)}$ .

Since

$$\epsilon_{k_2 N, k_1 M}^{R_1} + \epsilon_{k_1 N, k_2 M}^{R_2} = \left( \frac{k_2 N}{2R_1 \sin \theta_+} \right)^2 + \left( \frac{k_1 R_1 M}{\cos \theta_+} \right)^2, \quad (3.9)$$

the sum over  $N, M$  gives a product of the theta function  $\vartheta_3$ . The remaining  $P_n$  are evaluated as follows. Introducing a  $4K$ -vector  ${}^t \vec{v} = (\text{Re } z_n^{(1)}, \text{Im } z_n^{(1)}, \text{Re } w_n^{(1)}, \text{Im } w_n^{(1)}, \dots)$ , the exponent in  $P_n$  is expressed as  $-{}^t \vec{v} \cdot M_K \cdot \vec{v}$ , where  $M_K$  is a  $4K \times 4K$  symmetric matrix

$$M_K = \begin{pmatrix} 1_4 & C & & \dots & {}^t C \\ {}^t C & 1_4 & C & & \dots \\ & {}^t C & 1_4 & C & \dots \\ & & & \ddots & \\ C & & & & {}^t C & 1_4 \end{pmatrix} \quad \text{with } C = \begin{pmatrix} a(1_2 - \sigma^2) & 0 \\ b \cdot 1_2 & a(1_2 + \sigma^2) \end{pmatrix}. \quad (3.10)$$

$1_n$  is the  $n \times n$  unit matrix,  $\sigma^2$  is a Pauli matrix, and  $a = -s^2 q^{2n} D_n / 2$ ,  $b = c q^n (1 - q^{2n}) D_n$ . Performing the Gaussian integrals then gives  $P_n = D_n^K [\det M_K]^{-1/2}$ . The determinant of  $M_K$  here is regarded as a generalization of the circular determinant (see e.g. [25]), and obtained similarly:

$$\begin{aligned} \det M_K &= \prod_{k=1}^K \det(1 + \omega_k C + \omega_k^{-1} {}^t C) \\ &= D_n^{2K} \prod_{k=1}^K [1 - 2(c^2 + d_k s^2) q^{2n} + q^{4n}]^2 \\ &= D_n^{2K} [(p_n^+)^K - (p_n^-)^K]^4, \end{aligned} \quad (3.11)$$

where  $\omega_k = e^{2\pi i k / K}$ ,  $d_k = \cos(2\pi k / K)$  and  $p_n^\pm = (1/2) [\sqrt{1 - 2(c^2 - s^2) q^{2n} + q^{4n}} \pm (1 - q^{2n})]$ . To derive the last line, we have used a formula (A.1). The above expression shows

that  $\det M_K$  and hence  $Z(K)$  are actually analytic in  $K$ . We also notice that  $\prod P_n$  gives rise to a product of the theta function  $\vartheta_1$ . With the help of a formula (A.2), we finally obtain

$$Z(K) = g_+^{2K} |s|^{K-1} K \vartheta_3 \left( \frac{itKk_2^2}{2R_1^2 \sin^2 \theta_+} \right) \vartheta_3 \left( \frac{2itKk_1^2 R_1^2}{\cos^2 \theta_+} \right) \eta^{K-3}(2it) \prod_{k=1}^{K-1} \vartheta_1^{-1}(\nu_k | 2it), \quad (3.12)$$

for  $k_1 k_2 \neq 0$ , where  $\eta(\tau)$  is the Dedekind eta function, and

$$\pi \nu_k = \arcsin \left( |s| \sin \frac{\pi k}{K} \right). \quad (3.13)$$

The  $\vartheta_1$  part is similar to the oscillator part of the amplitude between D-branes at angles. This is naturally understood, once we notice that, in the  $K$ -sheeted Riemann surface  $\mathcal{R}_K$ , the array of the interfaces resembles pairs of D-branes at angles.

### 3.2 Case of $I_{12}^-$

The case of  $I_{12}^-$  is similar. For the oscillator part, it turns out that the integral expression for  $P_n$  is obtained by replacing in (3.8)  $c = \cos 2\theta_+$ ,  $s = \sin 2\theta_+$  with  $-\cos 2\theta_-$ ,  $\sin 2\theta_-$ , and thus the final expression by  $\theta_+ \rightarrow \theta_-$ , i.e.,  $R_1 \rightarrow 1/2R_1$ . For the zero-mode part, the expression corresponding to (3.9) is also obtained by  $R_1 \rightarrow 1/2R_1$ . Therefore,  $Z(K)$  in this case is obtained from (3.12) by  $R_1 \rightarrow 1/2R_1$  (and hence  $\theta_+ \rightarrow \theta_-$ ). This is expected, since  $I_{12}^-$  is constructed from the boundary state in which  $\text{CFT}_1$  is T-dualized compared with the boundary state for  $I_{12}^+$ .

### 3.3 Special cases

So far, we have considered the case of  $k_1 k_2 \neq 0$ . When  $k_1 k_2 = 0$ , while the analysis of the oscillator part remains the same, the zero-mode structure and the product of  $G_{12}^\pm$  and  $(G_{12}^\pm)^\dagger$  change. Repeating similar computations, one then finds for  $I_{12}^+$  that the product of  $\vartheta_3$ 's in (3.12) is replaced by  $\Theta_1^K \equiv \vartheta_3^K(itk_1^2/2R_2^2) \vartheta_3^K(2itk_1^2 R_1^2)$  for  $k_1 \neq 0, k_2 = 0$  and  $\Theta_2^K \equiv \theta_3^K(itk_2^2/2R_1^2) \theta_3^K(2itk_2^2 R_2^2)$  for  $k_1 = 0, k_2 \neq 0$ . When both  $k_1$  and  $k_2$  vanish, the original boundary states and hence the interfaces are not well-defined, since Cardy's condition is not satisfied. We will not discuss this case. From (2.8), one also finds that  $k_1 k_2 = 0$  implies  $s = 0$  (unless taking the decompactified limit (or its T-dual)  $R_{1,2} = 0, \infty$  which is not covered in our setting).  $Z(K)$  is then simplified as  $Z(K) = g_+^{2K} \Theta^K \eta^{-2K}(2it)$ , where  $\Theta = \Theta_1$  or  $\Theta_2$ . The results for  $I_{12}^-$  is obtained by  $R_1 \rightarrow 1/2R_1$  as above.

## 4. Entanglement entropy

Given the partition functions with the interfaces inserted, we would now like to discuss the entanglement entropy. In the following, we concentrate on the case of  $I_{12}^+$ , since the results for  $I_{12}^-$  are easily read off from those for  $I_{12}^+$ . We also focus on the case with  $k_1 k_2 \neq 0$ , unless otherwise stated.

To compute the entropy via (2.4), we need the analytic form of  $Z(K)$  in  $K$ . A way to obtain it is to continue the product in (3.12) with respect to  $K$ , and another is to use



the last expression in (3.11) in terms of  $p_n^\pm$ . We first adopt the former with the help of the Bernoulli polynomials and numbers. This provides a rather general method to carry out the replica trick. We then use the latter, which is more straightforward. In the course of showing the equivalence of the results from the two approaches, we find that the entropy is expressed by the dilogarithm function.

We start with the result of  $Z(K)$  in (3.12). Since the modular parameter  $t = \pi/(2 \log L)$  is small for  $L \gg 1$ , it is convenient to evaluate it by the modular transformation  $\tau \rightarrow -1/\tau$ . One then finds that (when  $|s| \neq 0$ )

$$Z(K) = \frac{(g_+^2 |s|)^K}{|k_1 k_2|} e^{-(K-3)\pi/24t} e^{\varphi(K)/t} \left(1 + \mathcal{O}(e^{-\mu/t})\right), \quad (4.1)$$

where  $\mu$  is a positive constant and

$$\varphi(K) = \frac{\pi}{2} \sum_{k=1}^{K-1} \left(\frac{1}{2} - \nu_k\right)^2, \quad (4.2)$$

with  $0 < \nu_k < 1$  ( $k = 1, \dots, K-1$ ). Note that  $\varphi(K)$  is of the form

$$\varphi(K) = \sum_{k=1}^{K-1} f\left(\frac{k}{K}\right), \quad f(x) = \frac{1}{2\pi} \arccos^2(|s| \sin \pi x). \quad (4.3)$$

Since  $f(x)$  is analytic around  $x = 0$ , we expand it as  $f(x) = \sum_{m=0}^{\infty} f_m x^m$ . A useful fact here is that  $\sum_k k^m$  is expressed by the Bernoulli polynomials  $b_n(x)$  and numbers  $b_n$  as in (A.4). From this and properties of  $b_n(x), b_n$  summarized in the appendix, it follows that

$$\begin{aligned} \partial_K \varphi(K) \Big|_{K=1} &= \sum_{m=0}^{\infty} \frac{f_m}{m+1} \partial_K b_{m+1}(K) \Big|_{K=1} \\ &= f(0) + \frac{1}{2} f'(0) + \int_0^{\infty} \frac{i f'(ix) - i f'(-ix)}{1 - e^{2\pi x}} dx. \end{aligned} \quad (4.4)$$

After plugging the explicit form of  $f(x)$  and changing the variables as  $u = \operatorname{arcsinh}(|s| \sinh \pi x)$ , we apply the result to (2.4), and obtain

$$S = \sigma(|s|) \log L - \log |k_1 k_2|, \quad (4.5)$$

up to terms vanishing for  $L \gg 1$ , where

$$\sigma(|s|) = \frac{|s|}{2} - \frac{2}{\pi^2} \int_0^{\infty} u \left( \sqrt{1 + (|s|/\sinh u)^2} - 1 \right) du. \quad (4.6)$$

We find that the entropy has a logarithmic scaling with respect to the size of the system  $L$ , but the coefficient  $\sigma(|s|)$  is a function of  $|s|$ . It turns out shortly that  $\sigma(|s|)$  is expressed by the dilogarithm function. The sub-leading term counts the product of the winding numbers. This is analogous to the topological entanglement entropy in (2+1)-dimensional systems characterizing the topological order [17, 18]. We also note that the entropy is a function of  $\theta_+, k_1, k_2$  only, and does not depend on  $\alpha, \beta, R_1, R_2$  explicitly.

In special cases,  $\sigma(|s|)$  is simplified. First, let us consider the topological case,  $|s| = 1$ . Since the identity is included as a special case, this case should reproduce the universal scaling of the ordinary entanglement entropy without interfaces. In fact, one finds that  $\sigma(1) = c/3 = 1/3$ , which agrees with the result [15, 16]. Next, when  $|s|$  is small, one can show that the second term in (4.6) is  $\mathcal{O}(|s|^2 \log |s|)$ , and that  $\sigma(|s|) \rightarrow |s|/2$ . This implies that the leading term decreases as  $|s|$  does, which also agrees with the fact that the oscillator part of the two CFT's are decoupling as  $|s| \rightarrow 0$ . The result for small  $|s|$  is derived also by directly expanding  $\nu_k$  in (3.13). For general  $|s|$ , one can check that  $\sigma(|s|)$  monotonically interpolates these two cases. This supports an intuition that the entanglement changes according to  $|s|$ , since  $|s|$  is the strength of the interaction between the two CFT's. In [9], certain reflection and transmission coefficients are introduced as probes of conformal interfaces. For the  $c = 1$  permeable interfaces, they give  $c^2$  and  $s^2$ . Compared with those coefficients, one finds that the entanglement entropy (4.5) can probe a little more details of the interfaces.

When the sizes of the two systems are  $L$  and  $\Lambda - L$ , the entanglement entropy without interfaces scales as  $(c/3) \log [(\Lambda/\pi) \sin(\pi L/\Lambda)] + \text{const.}$  [15], where the sub-leading constant term is independent of  $L$  [16]. In our case with interfaces, the entropy should also be symmetric under the exchange of the two CFT's, and the above scaling should be reproduced in a special case. A possible form for  $L \neq \Lambda/2$  satisfying these requirements is  $S = \sigma(|s|) \log [(\Lambda/2) \sin(\pi L/\Lambda)] - \log |k_1 k_2|$ .

Here, some comments for special cases may be in order. When  $k_1 k_2 = 0$  and hence  $s = 0$ , it follows from the result in section 3.3 that the entropy exactly vanishes:  $S = 0$ . This confirms the fact that the two CFT's decouple in this case. When  $k_1 k_2 \neq 0$  and hence  $s \neq 0$  (unless in the decompactified limit), the entropy might appear to be negative for small enough  $s$ . This, however, is not the case: To obtain (4.1) by modular transformations, we have used  $t/s^2 \ll 1$  for  $\vartheta_3$ 's. Thus, the result in (4.5) is valid when the first term is large enough. In fact, since  $L$  is the cutoff in our setting and can be arbitrarily large independently of other parameters, this condition is always satisfied by taking large enough  $L$ . Note that, however small  $s$  is, the two CFT's couple through the zero-modes if  $k_1 k_2 \neq 0$ . In order to analyze the case where  $k_1 k_2 \neq 0$  and  $s \log L$  is small with actually finite  $L$ , one may need to develop a method to compute the entanglement entropy for finite systems, e.g., by generalizing the results in [16].

One can also derive the entropy by applying the expression of  $Z(K) = g_+^{2K} \vartheta_3 \vartheta_3 q^{-K/6} \prod P_n$  in terms of  $p_n^\pm$ . Recalling the formula (2.4), we first evaluate

$$\begin{aligned} \sum_{n=1}^{\infty} \partial_K \log P_n \Big|_{K=1} &= -2 \sum_{n=1}^{\infty} F(2\pi t n) \\ &\simeq \frac{-1}{\pi t} \int_0^{\infty} F(y) dy + F(0), \end{aligned} \tag{4.7}$$

as  $t \rightarrow 0$ , where

$$F(y) = \log |s| - y + \sqrt{1 + (|s|/\sinh y)^2} \operatorname{arcsinh}(\sinh y/|s|), \tag{4.8}$$

and  $F(0) = 1 + \log |s|$ . We have used  $p_n^+ p_n^- = |s|^2 q^{2n}$ , and assumed that  $s$  is not vanishing so that  $F(0)$  is not divergent. In the case without the derivative  $\partial_K$ , a similar approximation by an integral is subtle, since the summand is singular at  $t = 0$ . Thus, we instead note that  $\prod_{n=1}^{\infty} P_n \big|_{K=1} = e^{-\pi t/3} \eta^{-2}(2it)$ , which after a modular transformation gives  $\sum_{n=1}^{\infty} \log P_n \big|_{K=1} \simeq \pi/(12t) + \log(2t)$ . Together with  $\vartheta_3 \vartheta_3 \simeq |s|/(2Kt|k_1 k_2|)$  for small  $t$ , the entropy is obtained as  $S = \tilde{\sigma}(|s|) \log L - \log |k_1 k_2|$ , with

$$\tilde{\sigma}(|s|) = \frac{1}{6} + \frac{2}{\pi^2} \int_0^{\infty} F(y) dy. \quad (4.9)$$

Compared with the previous result (4.5),  $\tilde{\sigma}(s)$  should agree with  $\sigma(s)$ . To show this, we consider their derivatives:

$$\sigma'(s) = \frac{1}{2} - \frac{2}{\pi^2} \int_0^{\infty} \frac{dw \operatorname{arcsinh} w}{w \sqrt{1+w^2} \sqrt{1+w^2/s^2}}, \quad \tilde{\sigma}'(s) = \frac{2}{\pi^2} \int_0^{\infty} \frac{dz \operatorname{arcsinh} z}{z \sqrt{1+z^2} \sqrt{1+s^2 z^2}}, \quad (4.10)$$

and

$$\sigma''(s) = \frac{-2}{\pi^2 s} \int_0^{\infty} \frac{dw (w/s^2) \operatorname{arcsinh} w}{\sqrt{(1+w^2)(1+w^2/s^2)^3}}, \quad \tilde{\sigma}''(s) = \frac{-2}{\pi^2 s} \int_0^{\infty} \frac{dz (s^2 z) \operatorname{arcsinh} z}{\sqrt{(1+z^2)(1+s^2 z^2)^3}}, \quad (4.11)$$

where we have made changes of variables  $w = \sinh u$  and  $z = s^{-1} \sinh y$ . The integral for  $\sigma''(s)$  here is performed as

$$\begin{aligned} -\frac{\pi^2 s}{2} \sigma(s)'' &= \frac{1}{s - 1/s} \left( \sqrt{\frac{w^2+1}{w^2+s^2}} \operatorname{arcsinh} w - \operatorname{arcsinh} \frac{w}{s} \right) \Big|_{w=0}^{\infty} \\ &= \frac{\log s}{s - 1/s}. \end{aligned} \quad (4.12)$$

One then finds that  $-(\pi^2 s/2) \tilde{\sigma}(s)''$  is also given by the above, namely  $\sigma(s)'' = \tilde{\sigma}(s)''$ , since the integral representations of  $s \cdot \sigma''$  and  $s \cdot \tilde{\sigma}''$  are related by  $s \leftrightarrow 1/s$ . It is easy to confirm that the integration constants are also the same, e.g., by checking special values  $\sigma(1) = \tilde{\sigma}(1) = 1/3$ ,  $\sigma'(1) = \tilde{\sigma}'(1) = 1/4$ , which verifies  $\sigma(s) = \tilde{\sigma}(s)$ .

As a by-product, we find by integrating  $\sigma''(s)$  that  $\sigma(s)$  is expressed as

$$\sigma(s) = \frac{1}{6} + \frac{s}{3} + \frac{1}{\pi^2} \left[ (s+1) \log(s+1) \log s + (s-1) \operatorname{Li}_2(1-s) + (s+1) \operatorname{Li}_2(-s) \right], \quad (4.13)$$

where  $\operatorname{Li}_2(z)$  is the dilogarithm function. We summarize some properties of  $\operatorname{Li}_2(z)$  in the appendix. Using them, one can rederive the values of  $\sigma(1)$ ,  $\sigma'(1)$ , and the small- $s$  behavior of  $\sigma(s)$ .

## 5. Summary

We have obtained the partition functions with the  $c = 1$  permeable interfaces inserted, and the entanglement entropy of the corresponding interface CFT analytically. The entropy scales logarithmically with respect to the size of the system, as in the case without interfaces [15, 16]. Its coefficient, however, is not a constant but a monotonic function of  $|s|$

controlling the permeability, and is given explicitly in terms of the dilogarithm function. The sub-leading term of the entropy counts the product of the winding numbers. This is analogous to the topological entanglement entropy, which characterizes the topological order in  $(2 + 1)$ -dimensional systems [17, 18].

Our results show that the entanglement entropy is a useful probe to the system, as in the case without interfaces. It would be interesting to study how general our findings are: For example, does the entropy always contain the topological information of the system? Does it always show the scaling as in the case without interfaces? Regarding such studies, it would be useful to generalize the analysis based on the conformal symmetry [15, 16, 24] to the case with interfaces. A complication with interfaces is that one has to keep track of the shape of interfaces under conformal transformations. It would also be interesting to consider implications of the entanglement entropy in the context of condensed matter physics and string theory.

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### A. Useful formulas

In the main text, we apply the formulas

$$\prod_{r=0}^{n-1} \left[ x^2 - 2xy \cos \left( \theta + \frac{2r\pi}{n} \right) + y^2 \right] = x^{2n} - 2x^n y^n \cos n\theta + y^{2n}, \quad (\text{A.1})$$

$$\prod_{r=1}^{n-1} \sin \left( \frac{r\pi}{n} \right) = \frac{n}{2^{n-1}}. \quad (\text{A.2})$$

We also use the Bernoulli polynomials (see e.g. [25])  $b_n(x)$  ( $n = 0, 1, 2, \dots$ ) defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi). \quad (\text{A.3})$$

Their derivatives are  $b'_n(x) = nb_{n-1}(x)$ . At  $x = 0$ , they give the Bernoulli numbers  $b_n$ , namely,  $b_n = b_n(0)$ .  $b_n$  with odd index vanish except for  $b_1$ , and  $b_0 = 1, b_1 = -1/2, b_2 = 1/6, b_4 = -1/30, \dots$ . One also has  $b_n = b_n(1)$  for  $n \neq 1$ , and  $b_1(1) = 1/2$ . By the Bernoulli polynomials, the sums of powers of natural numbers are expressed as

$$(m + 1) \sum_{k=1}^{n-1} k^m = b_{m+1}(n) - b_{m+1} \quad (n, m = 1, 2, \dots). \quad (\text{A.4})$$

The Bernoulli numbers with even index have an integral representation,

$$b_{2n} = 4n(-1)^n \int_0^\infty \frac{t^{2n-1}}{1 - e^{2\pi t}} dt. \quad (\text{A.5})$$

From this, it follows that

$$\frac{1}{m+1} \partial_n b_{m+1}(n) \Big|_{n=1} = \delta_{m,0} + \frac{1}{2} \delta_{m,1} + (i^m - (-i)^m) \int_0^\infty \frac{mt^{m-1}}{1 - e^{2\pi t}} dt. \quad (\text{A.6})$$

In section 4, we use the dilogarithm function defined by

$$\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = - \int_0^z \frac{\log(1-w)}{w} dw. \quad (\text{A.7})$$

$\sigma''(s)$  in (4.12) is integrated by using the above integral representation and

$$\int dz \text{Li}_2(z) = z \text{Li}_2(z) - z - (1-z) \log(1-z). \quad (\text{A.8})$$

From  $\text{Li}_2(1) = \pi^2/6$ ,  $\text{Li}_2(-1) = -\pi^2/12$ , one can check the values of  $\sigma(1)$  and  $\sigma'(1)$ . To derive the small- $s$  behavior of  $\sigma(s)$ , useful formulas are (A.7) and

$$\text{Li}_2(1-z) = -\text{Li}_2(z) - \log z \log(1-z) + \frac{\pi^2}{6}. \quad (\text{A.9})$$

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